University of California, Berkeley
Physics H7C Fall 2002   (*Strovink*)

FINAL EXAMINATION

Directions: Do all six problems, which have unequal weight. This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You may use a calculator. Laptops and palmtops should be turned off. Use at least one bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (35 points).
An excited nucleus $(A, Z)'$, initially at rest, undergoes the decay

$$(A, Z)' \to (A, Z) + \gamma \, ,$$

where (A, Z) is the ground state of the nucleus. (A, Z) has mass M and $(A, Z)'$ has mass $M + \Delta M$, where $\Delta M \ll M$.
(a) (20 points)
Taking into account to lowest order the recoil of the nucleus, compute the energy E_γ of the emitted gamma ray. Specifically, your answer should take the form

$$E_\gamma \approx C_1 \, \Delta M \left(1 - C_2 \frac{\Delta M}{M} \right) \, ,$$

where C_1 and C_2 are for you to supply.
(b) (15 points)
If the mean life τ of the excited nucleus is sufficiently short, it will be possible for the emitted gamma ray to be reabsorbed by a different ground-state nucleus that is initially at rest, via the reaction

$$\gamma + (A, Z) \to (A, Z)' \, .$$

Roughly how short does τ need to be?

Problem 2. (30 points).
A *twisted nematic cell* is the basic building block of a a liquid crystal display (LCD). An ideal twisted nematic cell with parameter α is described by the Jones matrix

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \, .$$

In *words*, describe what it does to
(a) (4 points)
Light that is plane polarized in the $\hat{x}$ direction.
(b) (4 points)
Light that is plane polarized in the direction $\hat{x}\cos\alpha + \hat{y}\sin\alpha$.
(c) (4 points)
Right-hand circularly polarized light.
(d) (4 points)
Left-hand circularly polarized light.
(e) (14 points)
In an ideal LCD, an unpolarized beam of light passes through a sandwich consisting of a vertical linear polarizer, a twisted nematic cell with parameter $\alpha = \pi/2$, and a horizontal linear polarizer. Then it reflects off a mirror and passes back through the same sandwich. If all elements are ideal, what fraction of the incident beam irradiance emerges from the display?

Problem 3. (30 points).
If one wants to coat a block of glass so that none of the (monochromatic) light that is normally incident upon it is reflected from the front surface, one chooses an optical coating such that

$$l_1 = \lambda_1/4$$
$$n_1 = \sqrt{n_0 n_2} \, ,$$

where l_1 is the coating's thickness, n_1 is the coating's refractive index, λ_1 is the wavelength of light *inside the coating*, and n_0 and n_2 are the refractive indices of the air and the glass, respectively. (For this statement to be strictly correct, the magnetic permeabilities of all three media should be the same.)

Now consider a nonrelativistic particle of mass m, travelling in one dimension toward positive x, that is incident on a rectangular potential barrier at $x = 0$. That is,

$$V(x < 0) = 0$$
$$V(x > 0) = V_0 ,$$

where V_0 is a real positive constant. Choose the particle's energy E to be larger than V_0. Classically, the particle will surmount the barrier. Quantum mechanically, with nonzero probability the particle will be reflected.

By analogy with the optical coating, show how to "coat" the potential barrier so that the particle will have exactly zero probability of being reflected. You may change neither the initial energy E nor the final potential V_0, nor may you use potential segments that are anything but rectangular in shape. Your answer should take the form of a sketch of the modified barrier. Please provide quantitative values for all relevant dimensions in your sketch, both length(s) and energies, expressed in terms of E, V_0, m, and fundamental constants.

Problem 4. (35 points).
When the potential $V(r)$ vanishes within a spherically symmetric region, the time-independent Schroedinger equation for a particle of mass M may be written

$$\left(-\frac{\hbar^2}{2M}\nabla_r^2 + \frac{L^2}{2Mr^2} \right) u(\vec{r}) = E\,u(\vec{r}) ,$$

where ∇_r^2 is the radial part of ∇^2 and $\vec{L}$ is the angular momentum operator. With the substitution

$$u(\vec{r}) \equiv r^{-1}\,\mathcal{W}(r)Y_{lm}(\theta,\phi) ,$$

the T.I.S.E. becomes

$$\left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial r^2} + \frac{L^2}{2Mr^2} \right)\mathcal{W}\,Y_{lm} = E\,\mathcal{W}\,Y_{lm} .$$

(a) (20 points)

Consider the case in which $l > 0$, and focus on very small r such that

$$\frac{\hbar^2}{Mr^2} \gg E .$$

In this region, show that, to a good approximation,

$$\mathcal{W}(r) \propto r^b ,$$

where b depends on l; find b.
(b) (15 points)
For this spherically symmetric potential, considering the case $l = 0$ as well as $l > 0$, which (if any) of the eigenfunctions $u(\vec{r})$ are likewise spherically symmetric? Azimuthally symmetric? Same questions for the *probability density* associated with these eigenfunctions? Explain your reasoning.

Problem 5. (40 points).
Consider a number $2N_0$ of spin-$\frac{3}{2}$ identical fermions (N_0 is even). These particles have mass m and move nonrelativistically in one dimension x. (Their motion in directions orthogonal to x doesn't vary in a way that is significant for this problem, and may be ignored.) The fermions do not have a significant mutual interaction, but they are under the influence of an external potential

$$V(x) = \tfrac{1}{2}m\omega_0^2 x^2 ,$$

where ω_0 is a real positive constant.
(a) (10 points)
What is the density of states $\Delta N/\Delta E$, where the possible values of E are the energy eigenvalues of the Hamiltonian?
(b) (10 points)
At temperature $T = 0$, taking $V = 0$ at $x = 0$, what is the energy of the most energetic fermion?
(c) (20 points)
Suppose instead that a single nonrelativistic particle of any kind is trapped in this same potential well (not necessarily at zero temperature). At $t = 0$ the expectation value $\langle x \rangle$ of its position is measured to be x_0 (not necessarily at the origin). What is the earliest time at which $\langle x \rangle$ must again be equal to x_0?

Problem 6. (30 points).

A satellite of small mass (compared to a star) is in circular orbit of radius R around an isolated star of mass M, which is at rest at the origin. It moves with a velocity $\ll c$ (which you need to compute). A laser on board the satellite emits a pulse of light which, when observed in the rest frame of the star, travels exactly in the radial direction away from the star, in the plane of the circular orbit. Eventually the light pulse is detected by an isolated observer who is at rest with respect to the star, but very far away from it compared to R.

By the time it is detected, the light pulse will have suffered a tiny redshift $\Delta\lambda_D$ due to the satellite's motion relative to the observer, plus a tiny redshift $\Delta\lambda_G$ due to the star's gravitational field. To lowest order in small quantities, calculate the ratio

$$\rho = \frac{\Delta\lambda_G}{\Delta\lambda_D} \ .$$